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STOCHASTIC SYSTEMS

On Reliability of Repairable Hot Double Redundant System with Arbitrarily Distributed Life- and Repair Times of Its Elements

V. V. Rykov^{*,a} **and N. M. Ivanova**^{**,b}

**Gubkin Russian State Oil and Gas University, Moscow, Russia*
***Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia*
e-mail: ^{a}*vladimir_rykov@mail.ru,* ^{b}*nm_ivanova@bk.ru*

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Abstract—The article introduces the concept of Marked Markov processes, which are used to study a repairable hot double redundant system with a single repair facility. It is assumed that components' life- and repair times follow arbitrary distributions. The proposed approach allows for calculating the system's reliability function, and its mean lifetime, as well as conducting the sensitivity analysis to the shape of input distributions. The new method was validated with numerical examples by comparing it with previously obtained analytical results and showed high accuracy.

Keywords: Marked Markov Processes, hot double redundant system, arbitrarily distributed life- and repair times, reliability, sensitivity analysis

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1. INTRODUCTION

Ensuring the reliability of systems, objects, and processes is one of the primary goals during their creation and subsequent operation. One of the methods for enhancing the structural reliability of a system is redundancy. It involves duplicating its critically important components or implementing higher-order redundancy. A system consisting of a primary element and one duplicate element is the simplest type of redundant system, and its study is intriguing from both theoretical and practical perspectives.

Redundant systems with various types of redundancy, as well as with the capability to restore failed elements, have been well-studied by many authors. Key findings regarding Markovian redundant systems, both repairable and non-repairable, can be found in classical works on mathematical reliability theory (see, for example, [1, 2], and others).

The extensive practical applications of redundant systems have led to the development of new mathematical models for more complex systems. For instance, in [3, 4], the reliability of a redundant system with heterogeneous elements, which have different lifetime distributions, is examined. The papers [5, 6] focus on systems with preventive and priority servicing of failed elements. The studies [7, 8] are dedicated to redundant systems with various types of system repair, where the failures of elements adhere to the Marshall-Olkin model. In the context of implementing redundancy in systems, it is also essential to consider optimization issues related to redundancy based on different criteria. Numerous studies have addressed this problem as well (see, for example, [9, 10]). Another challenge is determining the fundamental parameters of the system, specifically the shape of life- and repair times distributions of system elements.

Often in practice, the shape of life- and repair times distributions of system elements is unknown. In the earliest research on the reliability of redundant systems, assumptions were made regarding the exponential distribution of the corresponding random variables[1]. However, the emergence of new mathematical methods has enabled the examination of systems where one time follows the exponential distribution while the other one follows an arbitrary distribution. There exists a number of studies dedicated to the application of methods such as the introduction of supplementary variables, semi-Markov processes, and others to address these issues (see, for example, [11–14]). An extension of these studies is the sensitivity analysis of reliability metrics to the shape of initial distributions.

In [15], the study of a k-out-of-n system with exponentially distributed lifetime and arbitrary repair time distribution utilized the theory of decomposable semi-regenerative processes, which was previously proposed by V.V. Rykov as a generalization and development of Smith's regeneration idea [16]. Later, this approach was applied to investigate the reliability of cold redundant systems [17, 18], featuring arbitrary initial distributions for elements' life- and repair times. However, it turned out that this approach is not suitable for analyzing similar systems with hot redundancy and other more complex models.

In this article, a novel approach is proposed for the study of a repairable hot double redundant system, based on the new concept of Marked Markov processes (MMP).

The concept of MMP is introduced as an extension of the notion and theory of point processes [19]. In queuing theory, a point process is defined as an increasing sequence of non-negative random variables that describe the moments of occurrence of certain events along the positive half-axis of time [20]. Specific instances of point processes include the Poisson process and the Cox point process, where the times of event occurrences are governed by the exponential distribution. Another generalization of the Poisson process is the renewal process, in which the intervals between the occurrences of events follow a distribution that is different from the exponential one [21].

To describe a point process that possesses the Markov property, the concept of Markov point processes was introduced in 1977 [22]. Later, it was generalized to the concept of a Marked Markov point process, which is defined as a pair of random variables, where the first component is the time of occurrence of a random event, and the second is a mark, also a random variable, that additionally describes the corresponding event.

Since the introduction of the concept of point processes, the corresponding theory has been hotly developed, as it has found applications in various fields of science and engineering. The paper [23] is devoted to constructing a mathematical model for controlling automobile traffic at intersections using marked point processes. In [24], a multivariate point process is investigated for the problem of monitoring the state of a telecommunication connection. The applicability of spatial Poisson processes for modeling interactions between wireless devices is discussed in [25].

Introduced in this investigation, MMP differs from known concepts in that it consists of two components: the first represents the number of failed elements of the system, while the second one is a set of random marks that define their residual life- and repair times. Using the proposed approach, it is possible to compute the reliability function and the mean lifetime of a hot double redundant system with arbitrary life- and repair time distributions of its elements.

The article is organized as follows. The problem statement, key notations, and assumptions are provided in the next Section 2. Section 3 presents the transformation of marks over a separate regeneration period. In Section 4, the main characteristics of the model are obtained in terms of marks, and Section 5 proposes an algorithm for their computation. The validation of the proposed algorithm and examples of numerical analysis are presented in Section 6. Finally, the main results of the work are formulated, and directions for future research are outlined.

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[1] Following the classical works by A.Y. Khinchin, B.V. Gnedenko, and others, from now on, whenever distributions of times are mentioned, it is understood that these refer to distributions of the corresponding random variables.

2. PROBLEM STATEMENT. MMP

2.1. Problem Statement, Notations, and Assumptions

Let's consider a repairable hot double redundant system with a single recovery device for failing elements, denoted as $< GI_2|GI|1 >$. Here, the angle brackets <> indicate a closed system, and the symbols GI refer to a recurrent failure flow and a recurrent servicing mechanism with arbitrary distributions of life- and repair times at the first and second positions, respectively. The subscript index on the first symbol indicates the total number of elements in the system. The last position denotes the number of repair units. Introduce the following notations and assumptions:

- $A_i :$ $(i = 1, 2, \dots)$: lifetimes of system elements, which are assumed to be independent identically distributed (i.i.d.) random variables (r.v.) with common cumulative distribution function (c.d.f.) $A(t) = \mathbb{P}\{A_i \leqslant t\}$, probability density function (p.d.f.) $a(t) = A'(t)$, and finite mean time $\mu_a = \mathbb{E}[A_i] < \infty$ and variance $\sigma_a^2 = \mathbb{D}[A_i] < \infty$.
- $B_i :$ $(i = 1, 2, \dots)$: repair times of system elements, which are assumed to be i.i.d. r.v. with common c.d.f. $B(t) = \mathbb{P}\{B_i \leqslant t\}$, p.d.f. $b(t) = B'(t)$, and finite mean $\mu_b = \mathbb{E}[B_i] < \infty$ and variance $\sigma_b^2 = \mathbb{D}[B_i] < \infty$.

2.2. Definition of MMP

To study such a model, we introduce the concept of MMP with a discrete parameter,

$$Z(n) = \{(J(n), \mathbf{X}(n)),\ n = 0, 1, \dots\},$$

the first component of which is a conditional Markov chain $J(n) := J$ with at most countable state space $\mathcal{J}$, and the second is a set of random marks $\mathbf{X}(n) := \mathbf{X}_i(n)$, where $i \in \mathcal{J}$, with values in a measurable space $(E_i,\, \mathcal{E}_i)$. Such a process is determined by:

- transition probabilities $p_{ij}(\mathbf{X}_i)$ of the process J, which depend on the content of the mark $\mathbf{X}_i$ in state i;
- marks transformation operators $\Phi_{ij}(\mathbf{X}_i)$ for the transition from state i to state j of Markov chain and their distributions.

Remark 1. This process can be considered as a special type of Markov sequence with a common state space and specified using a transition kernel (process generator). However, for the study of various applications, it is more convenient to start from the concept of MMP.

Remark 2. Based on such a model, it is possible to study a wide class of processes, such as semi-Markov, semi-regenerative processes, Markov renewal processes, etc.

2.3. MMP Construction

Consider MMP construction on an example of $< GI_2|GI|1 >$ system. In this case, $J(n)$ represents the number of failed elements. As the marks $\mathbf{X}_i(n)$, we choose multidimensional r.v.'s whose contents are

— residual lifetime $X_0^{(1)}(n)$ of one element and residual repair time $X_0^{(2)}(n)$ of another element in state $i = 0$,

— residual lifetime $X_1^{(1)}(n)$ of one element and newly assigned repair time $X_1^{(2)}(n)$ of other one in state $i = 1$,

— residual repair time $X_2(n)$ of the element being under repair in state $i = 2$.

In this representation, the upper index indicates the serial number of the mark, the lower index indicates the system state as it was introduced above, and the variable in brackets stands for the step number.

Suppose that at the initial time (at step $n = 0$) both elements of the system are operational, $J(0) = 0$. Therefore, the initial content of the marks corresponds to the time to failure for each of

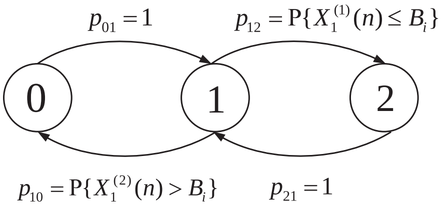

Fig. 1. Transition graph of the chain $J(n)$.

the system elements, $X_0^{(1)}(0) = A_0^{(1)}$, $X_0^{(2)}(0) = A_0^{(2)}$, where both $A_0^{(1)}$ and $A_0^{(2)}$ are i.i.d. r.v.'s with cdf $A(\cdot)$. Subsequently, the process finds itself in state $i = 0$ with a certain set of marks, the values of which will be calculated further.

The possible transitions of the considered chain and the marks transformations, describing the behavior of the system under consideration, are shown in Fig. 1.

From state 0, the process moves to state 1 regardless of the value of the marks with probability $p_{01}(\mathbf{X}(n)) = 1$. In this case, at the first step, the marks in state 1 are transformed as follows,

$$\mathbf{X}_1(1) = \Phi_{01}[\mathbf{X}_0(0)]: \quad X_1^{(1)}(1) = A_0^{(1)} \vee A_0^{(2)} - A_0^{(1)} \wedge A_0^{(2)}, \quad X_1^{(2)}(1) = B_1. \tag{1}$$

From state 1, transitions can occur to state 0 if the failed element is restored, or to state 2 if the second element fails. The probability of transitioning to state 0 in the first step is,

$$p_{10}(\mathbf{X}_1(1)) = \mathbb{P}\{X_1^{(1)}(1) > X_1^{(2)}(1)\}.$$

At that, the marks in state 0 are transformed as follows,

$$\mathbf{X}_0(1) = \Phi_{10}[\mathbf{X}_1(0)]: \quad X_0^{(1)}(1) = (X_1^{(1)}(1) - B_1)1_{\{X_1^{(1)}(1) > B_1\}}, \quad X_0^{(2)}(1) = A_1. \tag{2}$$

Similarly, when transitioning to state 2, the corresponding transition probability takes the form,

$$p_{12}(\mathbf{X}_1(1)) = \mathbb{P}\{X_1^{(1)}(1) \leqslant X_1^{(2)}(1)\},$$

then, the mark in state 2 is transformed as follows,

$$\mathbf{X}_2(1) = \Phi_{12}[\mathbf{X}_1(1)]: \ X_2(1) = (B_1 - X_1^{(1)}(1))1_{\{X_1^{(1)}(1) \leqslant B_1\}}. \tag{3}$$

Finally, from state 2, there is only one possible transition to state 1, so $p_{21}(\mathbf{X}_2(1)) = 1$. The marks transformation in the new state (at the second step) has the form,

$$\mathbf{X}_1(2) = \Phi_{21}[\mathbf{X}_2(1)]: \ X_1^{(2)}(2) = A_1, \quad X_1^{(1)}(2) = B_2. \tag{4}$$

Thus, it is clear that the transitions from state 2 to state 1 signify the regeneration of the system. Consequently, it suffices to focus on the system's behavior during a separate regeneration period.

3. BEHAVIOR OF THE PROCESS IN A SEPARATE REGENERATION PERIOD

Taking into account the regenerative nature of the system's behavior, consider the behavior of the process $Z(n)$ during a separate regeneration period. Similarly to the previous section, we will outline the transformation of marks during the regeneration period of the system. Denote $W(l)$ the residual lifetime of the system after failure of one of its elements at lth step,

$$W(l) = X_0^{(1)}(l) \vee X_0^{(2)}(l) - X_0^{(1)}(l) \wedge X_0^{(2)}(l) \ \ (l = 0, 1, 2, \dots).$$

Then, taking into account transformations of the marks (1)–(4) the following lemma is valid.

Lemma 1. *Between transitions from state 2 to state 1 (i.e. during the regeneration period), the marks are transformed as follows:*

$$\begin{aligned}
&X_0^{(1)}(l) = (X_1^{(1)}(l-1) - X_1^{(2)}(l-1))1_{\{X_1^{(1)}(l-1)>X_1^{(2)}(l-1)\}}, \quad X_0^{(2)} = A_l \in A(\cdot),\\
&X_1^{(1)}(l) = W(l-1), \quad X_1^{(2)} = B_l \in B(\cdot), \qquad\qquad (5)\\
&X_2(l) = (X_1^{(1)}(l) - X_1^{(2)}(l))1_{\{X_1^{(1)}(l)\leqslant X_1^{(2)}(l)\}},
\end{aligned}$$

where the initial contents of the marks are

$$X_0^{(1)}(0) = X_0^{(2)}(0) = 0, \quad X_1^{(1)}(0) = A_0, \quad X_1^{(2)}(0) = B_0, \quad X_2(0) = 0.$$

Proof. Indeed, with the initial marks defined as $X_1^{(1)}(0) = A_0$ and $X_1^{(2)}(0) = B_0$, the process goes directly to state 2 from state 1 if the failure of the operational elements occurs before the restoration of the repaired one, specifically when $A_0 \leqslant B_0$. In this scenario, the process will find itself in state 2 with the following mark:

$$X_2(1) = (X_1^{(1)}(0) - X_1^{(2)}(0))1_{\{X_1^{(1)}(0)\leqslant X_1^{(2)}(0)\}}.$$

From state 1, the process will move to state 0 if the repair of the failing element occurs before the failure of the working one, i.e., when $A_0 > B_0$; thus, the marks are transformed as follows:

$$X_0^{(1)}(1) = (X_1^{(1)}(0) - X_1^{(2)}(0))1_{\{X_1^{(1)}(0)>X_1^{(2)}(0)\}}, \quad X_0^{(2)}(1) = A_1.$$

Then the process will return to state 1 with probability 1 and the following marks,

$$X_1^{(1)}(1) = X_0^{(1)}(1) \vee A_1 - X_0^{(1)}(1) \wedge A_1 \equiv W(1), \quad X_1^{(2)}(1) = B_1.$$

From state 1 the process can go to state 2 with probability

$$\mathbb{P}\{X_1^{(1)}(1) \leqslant X_1^{(2)}(1)\} \equiv \mathbb{P}\{W(1) \leqslant B_1\}$$

and residual repair time (the mark)

$$X_2(2) = (X_1^{(2)}(1) - X_1^{(1)}(1))1_{\{X_1^{(1)}(1)\leqslant X_1^{(2)}(1)\}},$$

after which it will return to state 1 with probability 1. This will end the regeneration period.

Otherwise, the regeneration period will continue and the process will go to state 0 with probability

$$\mathbb{P}\{X_1^{(1)}(1) > X_1^{(2)}(1)\} \equiv \mathbb{P}\{W(1) > B_1\}$$

and marks

$$X_0^{(1)}(2) = (X_1^{(1)}(1) - X_1^{(2)}(1))1_{\{X_1^{(1)}(1)>X_1^{(2)}(1)\}}, \quad X_0^{(2)}(2) = A_2.$$

Similarly, the transformation of the marks at the lth step occurs. From state 1, the process will return to state 0 l times if, during the $l-1$th visit to state 1 from state 0, the repair of the element completes before the residual operational time of the working element expires,

$$X_1^{(2)}(l-1) = B_{l-1} < W(l-1) = X_1^{(1)}(l-1).$$

In this case, the residual time to failure will decrease by $X_1^{(2)}(l-1) = B_{l-1}$, and the content of the mark in state 0 transforms as

$$X_0^{(1)}(l) = (X_1^{(1)}(l-1) - X_1^{(2)}(l-1))1_{\{X_1^{(1)}(l-1) > X_1^{(2)}(l-1)\}} \equiv (W(l-1) - B_{l-1})1_{\{W(l-1)>B_{l-1}\}}.$$

The repaired element initiates a new time to failure given by $X_0^{(2)}(l) = A_l$. Thus, considering the contents of the marks, this transition confirms the first of the formulas stated in (5).

From state 0, the process will return to state 1 with probability 1. Then, the residual time to failure of the system $W(l)$ is defined as the difference between the maximum and minimum of the r.v.'s $X_0^{(1)}(l)$ and A_l, and the repair of the second element begins, i.e., the second mark takes the value B_l,

$$X_1^{(1)}(l) = X_0^{(1)}(l) \vee A_l - X_0^{(1)}(l) \wedge A_l \equiv W(l), \quad X_1^{(2)}(l) = B_l,$$

that proves the second of the formulas (5).

Finally, from state 1, in case of a failure of the working element, i.e.,

$$X_1^{(2)}(l) = B_l \geqslant W(l) = X_1^{(1)}(l)$$

the process goes to state 2 with the mark

$$X_2(l) = (X_l^{(1)}(l) - X_1^{(2)}(l))1_{\{X_1^{(1)}(l) \leqslant X_l^{(2)}(l)\}} \equiv (B_l - W(l))1_{\{W(l)\leqslant B_l\}},$$

that proves the last of the formulas (5) and completes the proof of the lemma. □

In the next section, the obtained expressions are used to calculate the main characteristics of the system reliability.

4. CALCULATION OF THE MAIN RELIABILITY CHARACTERISTICS

The main reliability characteristics of the considered system during the regeneration period with the initial value of the marks

$$X_0^{(1)}(0) = X_0^{(2)}(0) = 0, \quad X_1^{(1)}(0) = A_0, \quad X_1^{(2)}(0) = B_0, \quad X_2(0) = 0,$$

are:

- the distribution $\pi_l = \mathbb{P}\{\nu = l\}$ of the number of steps ν until the system failure;
- transition durations $T_{ij}(l)$, $(i, j = 0, 1, 2,\ l = 1, 2, \dots)$ for moving from state i to state j at the lth step and their characteristics;
- the time R from the moment the process returns from state 2 to state 1 until the next moment of transition to state 2, the corresponding distribution $R(t)$, i.e. the reliability function of the system, as well as the corresponding mean system lifetime μ_R;
- the distribution of the regeneration period of the system $\Pi = R + T_{21}(\nu)$.

Calculate further the main characteristics of the model in terms of the marks. Denote by

$$T(l) = T_{10}(l-1) + T_{01}(l), \quad l = 1, 2, \dots,$$

the time until returning to state 1 at the lth step of the regeneration period. Denote by Ω_l and $\hat{\Omega}_l$ the sets of elementary events for which the following relations are satisfied,

$$\Omega_l = \{\omega : W(0) > B_0, W(1) > B_1, \dots, W(l) > B_l\} \quad (l = 1, 2, \dots),$$
$$\hat{\Omega}_l = \Omega_{l-1} \cap \{W(l) \leqslant B_l\}, \ \hat{\Omega}_0 = \{A_0 \leqslant B_0\},$$

as well as the corresponding probability

$$p_0 = \mathbb{P}\{A_0 > B_0\}, \;\; p_i = \mathbb{P}\{W(i) > B_i \,|\, W(i-1) > B_{i-1}\}. \tag{6}$$

Thus, the following holds.

Theorem 1. *The distribution π_l of the number of steps ν during the regeneration period, transition durations $T_{ij}(l)$ in state i before transition to state j at the lth step, the system operating time R before its failure, its regeneration period Π in terms of the marks are expressed as follows:*

$$\pi_l = \mathbb{P}\{\nu = l\} = \mathbb{P}(\hat{\Omega}_l), \tag{7}$$

$$T_{10}(l) = X_1^{(2)}(l)1_{\{X_1^{(1)}(l) > X_1^{(2)}(l)\}}, \quad T_{01}(l) = X_0^{(1)}(l) \wedge X_0^{(2)}(l), \tag{8}$$

$$T(l) = T_{10}(l-1) + T_{01}(l), \tag{9}$$

$$T_{12}(l) = X_1^{(1)}(l)1_{\{X_1^{(1)}(l) \leqslant X_1^{(2)}(l)\}}, \quad T_{21}(l) = X_2(l), \tag{10}$$

$$R = \sum_{l \geqslant 0}[T(l) + T_{12}(l)]1_{\{\hat{\Omega}_l\}}, \tag{11}$$

$$\Pi = R + T_{21}(l)1_{\{\hat{\Omega}_l\}}. \tag{12}$$

At that, the condition for the finiteness of the regeneration period mean time is the convergence of the series,

$$\mathbb{P}\{\nu \geqslant l\} = \sum \mathbb{P}(\hat{\Omega}_l) = \sum_{l \geqslant 0}(1 - p_l)\prod_{i=0}^{l-1} p_i < \infty. \tag{13}$$

Proof. Indeed, according to lemma 1, starting from state 1 with the marks $X_1^{(1)}(0) = A_0$, $X_1^{(2)}(0) = B_0$, the process immediately goes to state 2 if the failure of the working element occurs before the restoration of the one being repaired, i.e., when $X_1^{(1)}(0) \leqslant X_1^{(2)}(0)$, having spent time in this state equal to

$$T_{12}(1) = X_1^{(1)}(0) \wedge X_1^{(2)}(0) = X_1^{(1)}(0)1_{\{X_1^{(1)}(0) \leqslant X_1^{(2)}(0)\}}.$$

In this case, the process will be in state 2 with the mark $X_2(1) = (B_0 - A_0)1_{\{A_0 \leqslant B_0\}}$ and will spend a random time in it equal to this value,

$$T_{21}(1) = (B_0 - A_0)1_{\{A_0 \leqslant B_0\}}.$$

Thus,

$$\pi_0 = \mathbb{P}\{A_0 \leqslant B_0\}, \;\; R = A_0, \;\; \Pi = B_0,$$

which corresponds to formulas (7), (10)–(12) for $l = 0$.

The number of returns to state 1 will be equal to 1 if the element being repaired is restored before the working element fails, i.e., when $A_0 > B_0$; in this case, the process will spend a random time in state 1 before transitioning to state 0, equaled to

$$T_{10}(1) = X_1^{(1)}(0) \wedge X_1^{(2)}(0)1_{\{X_1^{(1)}(0) > B_0\}} \equiv A_0 \wedge B_0 1_{\{A_0 > B_0\}}$$

and will go to state 0 with marks

$$X_0^{(1)}(1) = (A_0 - B_0)1_{\{A_0 > B_0\}}, \quad X_0^{(2)}(1) = A_1.$$

Then the process will return to state 1 with probability 1 and the marks

$$X_1^{(1)}(1) = X_0^{(1)}(1) \vee A_1 - X_0^{(1)}(1) \wedge A_1 \equiv W(1), \quad X_1^{(2)}(1) = B_1$$

during random time $T_{01}(1) = X_0^{(1)}(l) \wedge X_0^{(2)}(l)$.

Then, from state 1 the process will go to state 2 with probability $\mathbb{P}\{W(1) \leqslant B_1\}$ and the mark

$$X_2(1) = (B_1 - W(1))1_{\{W(1)\leqslant B_1\}},$$

spending time $T_{12}(1) = W(1)1_{\{W(1)\leqslant B_1\}}$ in it, after which with probability 1 during time $T_{21}(1) = (B_1 - W(1))1_{\{W(1)\leqslant B_1\}}$ it will return to state 1. This will end the cycle. Thus, we have

$$\begin{aligned}\pi_1 &= \mathbb{P}\{\nu = 1\} = \mathbb{P}(\hat\Omega_1),\\ R &= T_{10}(1) + T_{01}(1) + T_{12}(1),\\ \Pi &= R + T_{21}(1),\end{aligned}$$

which corresponds to the formulas (7)–(12) for $l = 1$.

Similarly, the number of returns to state 1 will be equal to l if the event $\hat\Omega_l$ occurs, i.e., if the process $l-1$ times goes from state 1 to 0 and back, and at the lth step goes to state 2. The probability of this event is,

$$\pi_l = \mathbb{P}\{\nu = l\} = \mathbb{P}(\hat\Omega_l),$$

which proves formula (7). In this case, the transformations of the marks are carried out by lemma 1, and transitions durations $T_{10}(l)$ from state 1 to 0 and back $T_{01}(l)$ at the lth step are equal to

$$\begin{aligned}T_{10}(l) &= X_1^{(2)}(l)1_{\{X_1^{(1)}(l)>X_1^{(2)}(l)\}}, \quad T_{01}(l) = X_0^{(1)}(l) \wedge X_0^{(2)}(l)1_{\{X_0^{(1)}(l)>X_0^{(2)}(l)\}},\\ T_{12}(l) &= X_1^{(1)}(l)1_{\{X_1^{(1)}(l)\leqslant X_1^{(2)}(l)\}}, \quad T_{21}(l) = X_2(l).\end{aligned}$$

The idea that the system's operating time R before failure consists of sequences of repair times and residual operating times before failure of elements, i.e., transitions durations between states 1 and 0 before transition to state 2, combined with the regeneration period Π that encompasses the time $T_{21}(l)$ spent in state 2, substantiates the validity of equations (11), (12).

Find the condition for the finiteness of the regeneration period mean time. Since the time of return from state 2 to state 1 is finite with probability 1 for any step of the cycle, it is sufficient that the system operates until failure with probability 1,

$$\mathbb{P}\{R < \infty\} = 1.$$

First, consider the condition of the finiteness of the number of steps in the regeneration period, $\mathbb{P}\{\nu < \infty\} = 1$. Noting that

$$\mathbb{P}\{\nu < \infty\} = \sum_{l\geqslant 0} \mathbb{P}\{\nu = l\} = \sum_{l\geqslant 0} \mathbb{P}(\hat\Omega_l),$$

calculate the probability of the event occurrence $\hat\Omega_l$ using the Markov dependence of the marks,

$$\begin{aligned}\mathbb{P}(\hat\Omega_l) &= \mathbb{P}\{W(l) \leqslant B_l \,|\, \Omega_{l-1}\}\mathbb{P}(\Omega_{l-1}) = \mathbb{P}\{W(l) \leqslant B_l \,|\, W(l-1) > B_{l-1}\}\mathbb{P}(\Omega_{l-1})\\ &= (1-p_l)\mathbb{P}(\Omega_{l-1}) = (1-p_l)\mathbb{P}(\Omega_{l-1}|\Omega_{l-2})\mathbb{P}(\Omega_{l-2}) = \cdots = (1-p_l)p_{l-1}\cdots p_0,\end{aligned}$$

where the notations (6) are used. Summing up the obtained expressions, we find

$$\sum_{l\geqslant 0} \pi_l = 1 - p_0 + p_0(1-p_1) + \ldots p_0 \cdots p_{l-1}(1-p_l) + \cdots = 1 - \prod_{l\geqslant 0} p_l.$$

Thus, the condition for the finiteness of the number of steps in the regeneration period is the divergence of the product $\prod_{l\geqslant 0} p_l$ or, equivalently, the divergence of the series $\sum\limits_{l\geqslant 0} \ln p_l$. Moreover, for the finiteness of the regeneration period mean time $\mathbb{E}[R] < \infty$ it is sufficient

$$\mathbb{E}[R] = \sum \mathbb{P}\{\nu > l\} = \sum \mathbb{P}(\hat\Omega_l) = \sum_{l\geqslant 0} (1-p_l) \prod_{i=0}^{l-1} p_i < \infty$$

which coincides with the formula (13) and completes the proof of the theorem. □

The obtained expressions show that the main characteristics of the model are expressed in terms of the marks. However, analytical expressions of their distributions and numerical indicators are quite cumbersome and require the introduction of special transformations. At the same time, the computational procedures for presenting the final results are quite problematic. Therefore, the next section proposes an algorithm for calculating the main characteristics of the model by simulation modeling, based directly on the proposed approach.

5. ALGORITHM FOR CALCULATING THE MAIN RELIABILITY CHARACTERISTICS

This section presents an algorithm for calculating the reliability characteristics of a repairable hot double redundant system based on the proposed method.

Algorithm 1.

Initial data:

$A(\cdot)$, $B(\cdot)$ – the distributions of r.v.'s A_i, B_i, corresponding means (μ_a, μ_b) and coefficients of variation (v_a, v_b),

N – total number of model realizations,

n – the current number of realizations,

ν_l – the array of the number of cycles length l during the regeneration period,

l – regeneration period length counter,

R – the array of times to system failure,

Π – system regeneration period array.

Input: $A(\cdot)$, $B(\cdot)$, N, $n = 1$, $\nu_l = [0] * \max(l)$, $R = [0] * N$, $\Pi = [0] * N$.

Beginning. Put: $l = 0$, $X_1^{(1)}(0) = A_0 \in A(\cdot)$, $X_1^{(2)}(0) = B_0 \in B(\cdot)$, $X_0^{(1)}(0) = X_0^{(2)}(0) = X_2(0) = 0$, $T(0) = 0$.

Step 1. If $n < N$, go to Step 2, if no, go to Step 4.

Step 2. While $X_1^{(1)}(l) > X_1^{(2)}(l)\ \forall\, l = 0, 1, \ldots$, repeat:

$$l := l + 1$$

$$T_{10}(l) = X_1^{(2)}(l-1)$$

$$X_0^{(1)}(l) = X_1^{(1)}(l-1) - X_1^{(2)}(l-1), \quad X_0^{(2)}(l) = A_l \in A(\cdot)$$

$$T_{01}(l) = X_0^{(1)}(l) \wedge X_0^{(2)}(l)$$

$$T(l) := T(l-1) + T_{10}(l) + T_{01}(l)$$

$$X_1^{(1)}(l) = X_0^{(1)}(l) \vee X_0^{(2)}(l) - X_0^{(1)}(l) \wedge X_0^{(2)}(l), \quad X_1^{(2)}(l) = B_l \in B(\cdot),$$

in another case $X_1^{(1)}(l) < X_1^{(2)}(l)$

$$T_{12}(l) = X_1^{(1)}(l)$$

$$T_{21}(l) = X_2(l) = X_1^{(2)}(l) - X_1^{(1)}(l).$$

Go to Step 3.

Step 3. Collect statistics:

- Filling the array $\nu_l := \nu_l + 1$,
- Filling the array $R_n := T(l) + T_{12}(l)$,
- Filling the array $\Pi_n := R_n + T_{21}(l)$.

Put $n := n + 1$ and go to the Beginning.

Step 4. Processing statistics:

- calculating of the estimate of the distribution of steps number ν during the regeneration period,

$$\hat{\pi} = \frac{\nu_l}{\sum_{l>0} \nu_l},$$

– calculating of the empirical reliability function R,

$$\hat{R}(t) = 1 - \frac{1}{N} R_n, \quad R_n \leqslant t \leqslant R_{n+1},$$

– calculating of the system mean lifetime,

$$\hat{\mu}_R = \frac{1}{N} \sum_{n=1}^{N} R_n,$$

– Results printing.

STOP.

6. EXAMPLES

In this section, we will consider several examples of numerical analysis, and compare the results obtained using the proposed approach and methods of Markov process theory. The choice of initial data does not refer to any example from a practical problem and is only demonstrative.

6.1. $< M_2|M|1 >$ *System*

Let life- and repair times of elements follow an exponential distribution with parameters α and β, respectively, $A(t) \sim Exp(\alpha)$, $B(t) \sim Exp(\beta)$. The reliability characteristics of such a system can be calculated by a direct method from the birth and death process. Thus, the reliability function of $< M_2|M|1 >$ system is calculated as

$$R(t) = \frac{1}{2r} e^{-\frac{1}{2}(3\alpha+\beta+r)t} \left(e^{rt}(\alpha+\beta+r) - (\alpha+\beta-r) \right), \tag{14}$$

where $r = \sqrt{\alpha^2 + 6\alpha\beta + \beta^2}$, from which we obtain the mean system lifetime μ_R,

$$\mu_R = \int_0^\infty R(t)dt = \frac{1 + 2\alpha\beta^{-1}}{2\alpha^2\beta^{-1}}. \tag{15}$$

To illustrate the example, suppose that $\alpha = \beta = 1$. In this case, the mean life- and repair times are $\mu_a = \mu_b = 1$. For the experiment, the number $N = 10^5$ of algorithm realizations is chosen experimentally to achieve high accuracy of the algorithm results. The direct implementation of the algorithm is performed using Python programming language. To assess the accuracy of the results

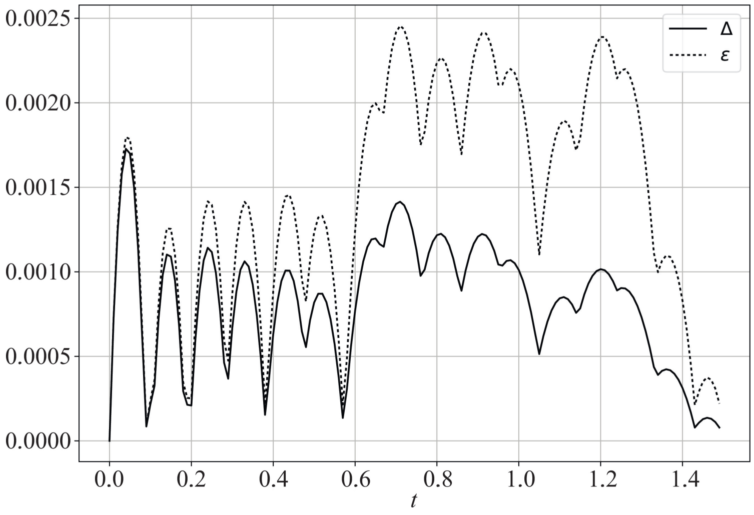

Fig. 2. Absolute and relative errors of the reliability function estimation for $< M_2|M|1 >$ system.

obtained using the proposed algorithm, we consider the absolute Δ and relative ε errors,

$$\Delta = |R_{th}(t) - \hat{R}(t)|, \quad \varepsilon = \frac{\Delta}{R_{th}(t)},$$

where $R_{th}(t)$ denotes the values of the reliability function, calculated by the formula (14), and $\hat{R}(t)$ presents the values obtained using the algorithm. Figure 2 shows the graphical results of calculating these estimates on the interval $t = [0, 1.5]$, where $t = 1.5$ corresponds to the mean lifetime of the system μ_R, calculated by the formula (15). It is evident from the graph that on the considered interval t the maximum value of the absolute error is $\Delta \approx 0.0017$, and the maximum value of the relative error is $\max(\varepsilon) \approx 0.0025$. At the same time, mean values of estimates on the entire interval t are equal to $\mathbb{E}[\Delta] = 0.0008$ and $\mathbb{E}[\varepsilon] = 0.0014$, respectively.

The values of the mean system lifetime μ_R, calculated analytically and using the algorithm, are equal to $\mu_R = 1.5$ and $\hat{\mu}_R = 1.502127$, respectively, and emphasize the comparability of the obtained results.

6.2. $< M_2|GI|1 >$ System

Consider further the case when lifetime of elements has an exponential distribution and repair time follows an arbitrary distribution, i.e., consider $< M_2|GI|1 >$ system.

To calculate the analytical expression for the reliability function of such a system, the method of introducing supplementary variables is used [26], which leads to the need to solve the system of Kolmogorov partial differential equations,

$$\frac{d}{dt}\pi_0(t) = -2\alpha\pi_0(t) + \int\limits_0^t \beta(x)\pi_1(t,x)dx,$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\pi_1(t;x) = -(\alpha + \beta(x))\pi_1(t,x),$$

$$\frac{d}{dt}\pi_2(t) = \alpha\int\limits_0^t \pi_1(t,x)dx,$$

jointly with initial $\pi_0(0) = 0$, $\pi_1(0) = 1$, $\pi_2(0) = 0$ and boundary conditions $\pi_1(t, 0) = 2\alpha\pi_0(t)$ using the method of characteristics. By solving this system in terms of Laplace transform (LT), LT of the reliability function is obtained, which has the following form in terms of LT of components repair time $\tilde{b}(s)$,

$$\tilde{R}(s) = \frac{s + \alpha(2 - \tilde{b}(s+\alpha))}{(s+\alpha)(s + 2\alpha(1 - \tilde{b}(s+\alpha)))}, \tag{16}$$

from where we calculate the mean system lifetime,

$$\mu_R = \tilde{R}(0) = \frac{2 - \tilde{b}(\alpha)}{2\alpha(1 - \tilde{b}(\alpha))}.$$

To compare the analytical results and their estimates obtained using the new algorithm, we assume that the repair time of the elements has a Gamma distribution ($\Gamma = \Gamma(l, \theta)$) with the following characteristics, i.e., p.d.f. $b(t) = \theta^l t^{l-1} e^{-\theta t}\Gamma(l)^{-1}$, $t > 0$, mean $\mu_b = l\theta^{-1} = 1$ and coefficient of variation $v_b = \sqrt{l}/l = 0.5$. For the remaining parameters, we again use $\alpha = 1$, $N = 10^5$. To evaluate the obtained results, we also consider the absolute Δ and relative ε errors (see Fig. 3). To calculate these estimates, the inverse LT of the reliability function (16), calculated numerically using the built-in function of Python programming language, is taken as the theoretical value $R_{th}(t)$. The argument t is considered on the interval $t = [0, 1.35]$, where $\max(t) = 1.35$ corresponds to μ_R for the given parameters.

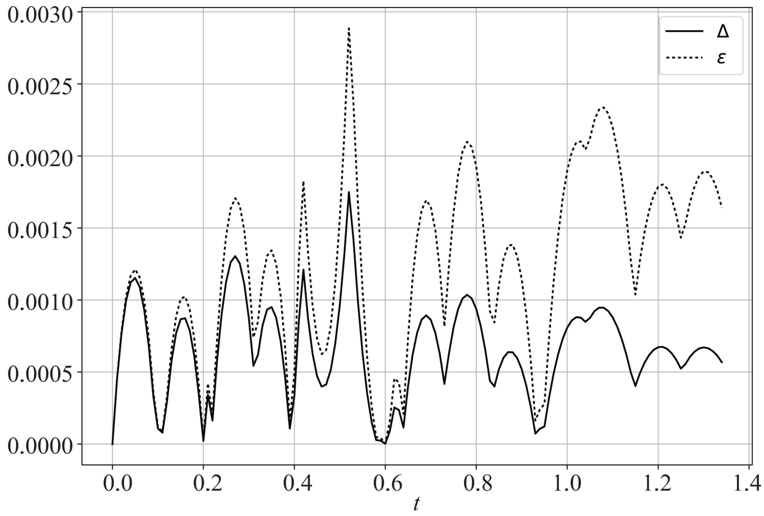

Fig. 3. Absolute and relative errors of the reliability function estimation for $< M_2|\Gamma|1 >$ system.

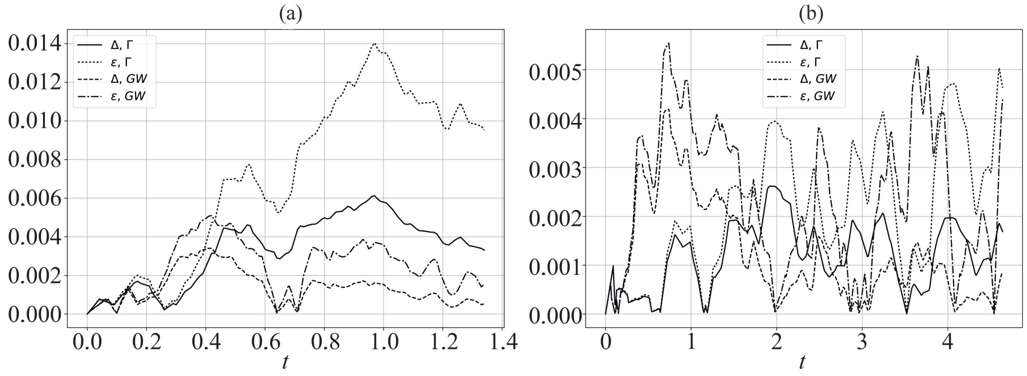

Fig. 4. Absolute and relative errors of the reliability function estimation for $< M_2|GI|1 >$ system. (a) — $v_b = 0.5$. (b) — $v_b = 5$.

According to the results of the experiment, the maximum value for Δ is $\max(\Delta) \approx 0.0017$, and for ε it is $\max(\varepsilon) \approx 0.0028$. The mean values over the entire interval t are $\mathbb{E}[\Delta] = 0.0007$ and $\mathbb{E}[\varepsilon] = 0.0012$. In this case, the mean system lifetime is $\mu_R = 1.3469$ for the analytical result and $\hat{\mu}_R = 1.3478$ for the algorithm. The results of the algorithm evaluation again demonstrate high accuracy relative to the analytical results.

For additional verification, we consider the simulation modeling of a repairable hot double redundant system using the discrete-event modeling (DEM) method [27]. The method is implemented using Python programming language, for which the total simulation time is taken as $T = 10^5$. As an example, consider Γ and Gnedenko-Weibull (GW) distributions of the repair time of elements, while fixing the mean $\mu_b = 1$ and the coefficient of variation $v_b = 0.5$, 5 for both distributions. The remaining parameters are the same as previously.

Figures 4a and 4b show the absolute and relative errors of estimating the reliability function of $< M_2|GI|1 >$ system.

Parameter t is chosen as the maximum value of the mean system lifetime for $v_b = 0.5$ and $v_b = 5$, respectively, for each figure. The results of the reliability function assessment using DEM method are taken as $R_{th}(t)$. The obtained results demonstrate that among the considered repair time distributions and the corresponding coefficients of variation, the maximum errors were $\max(\Delta) \approx 0.006$ and $\max(\varepsilon) \approx 0.014$.

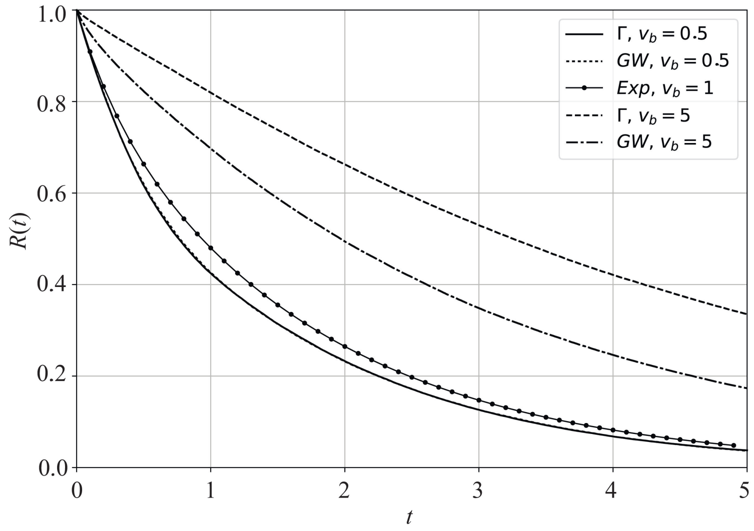

Fig. 5. The reliability function of $< M_2|GI|1 >$ system.

The mean values of these errors for all cases are presented in Table 1. The results of this experiment confirm the accuracy of the reliability characteristics obtained using the proposed approach.

Below is the graph of the reliability function of $< M_2|GI|1 >$ system obtained using the proposed approach (see Fig. 5). The legend of the figure defines the line type for the selected repair time distributions and the corresponding coefficients of variation. To identify the influence of the coefficient of variation of repair time on the reliability function, the figure also shows the case of the Markov model (*Exp*).

The results of this example show that the shape of repair time distribution and coefficient of variation significantly affect the reliability function. However, it is worth noting that the reliability function curves as $v_b = 0.5$ and $v_b = 1$ are located quite close to each other. Moreover, as $v_b = 0.5$, despite the different repair time distributions, these curves merge on the considered interval t. In the case of $v_b = 5$, on the contrary, the curves of the reliability function for the same repair time distributions differ significantly from each other. Thus, we can conclude that the reliability function is insensitive to the shape of repair time distribution for fixed mean μ_b and coefficient of variation v_b as $v_b < 1$ and, conversely, about its sensitivity as $v_b > 1$.

Table 2 presents the mean system lifetime for this example and also demonstrates the corresponding values obtained using DEM method. The results show the insensitivity of the mean μ_R to the shape of repair time distribution for fixed μ_b and v_b as $v_b < 1$. In addition, the mean system lifetime turns out to be sensitive to the shape of repair time distribution and coefficient of variation as $v_b > 1$.

Table 1. Mean errors Δ and ε

$B(\cdot)$, v_b	$\mathbb{E}[\Delta]$	$\mathbb{E}[\varepsilon]$
Γ, $v_b = 0.5$	0.0034	0.0070
Γ, $v_b = 5$	0.0006	0.0007
GW, $v_b = 0.5$	0.0014	0.0024
GW, $v_b = 5$	0.0023	0.0031

Table 2. Mean lifetime of $< M_2|GI|1 >$ system

$B(\cdot)$, v_b	DEM	Algorithm
Γ, $v_b = 0.5$	1.34619	1.34457
Γ, $v_b = 5$	4.63125	4.64059
GW, $v_b = 0.5$	1.34111	1.352305
GW, $v_b = 5$	2.80304	2.82017

6.3. $< GI_2|GI|1 >$ System

In the last example, we consider a system whose life- and repair time of elements have arbitrary distributions. We again take Γ and GW as these distributions. Denote v_a the coefficient of variation of components' lifetime and put $v_a = 0.5$. The remaining initial parameters are retained from the

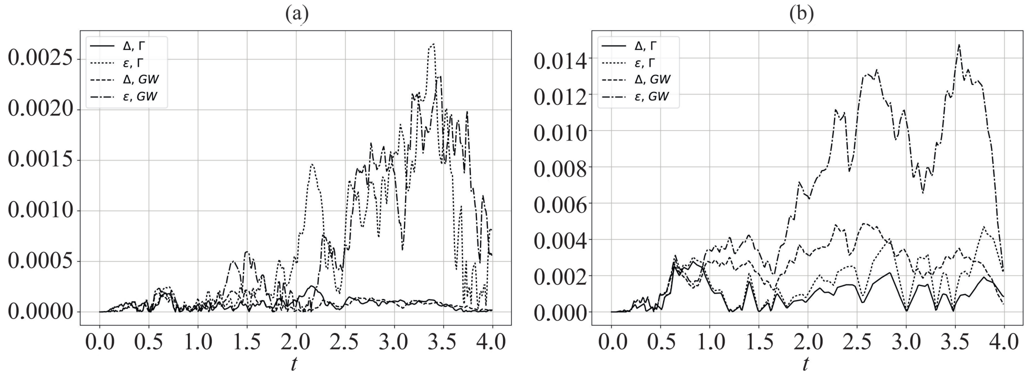

Fig. 6. Absolute and relative errors of the reliability function estimation for $< \Gamma_2|GI|1 >$ system. (a) — $v_b = 0.5$. (b) — $v_b = 5$.

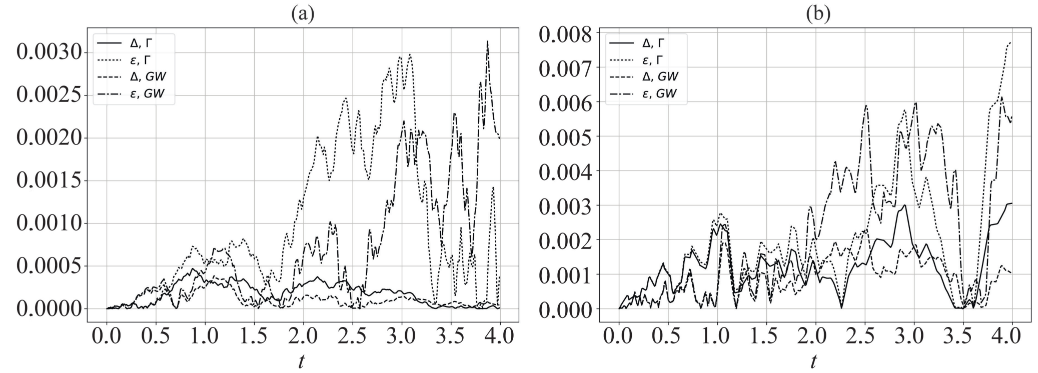

Fig. 7. Absolute and relative errors of the reliability function estimation for $< GW_2|GI|1 >$ system. (a) — $v_b = 0.5$. (b) — $v_b = 5$.

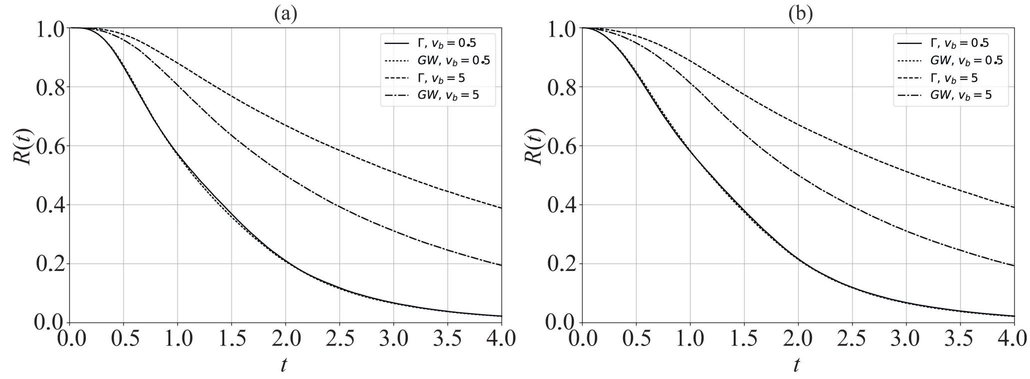

Fig. 8. The reliability function of $< GI_2|GI|1 >$ system. (a) — $A \sim \Gamma$, $v_a = 0.5$. (b) — $A \sim GW$, $v_a = 0.5$.

previous example. Figures 6 and 7 demonstrate the estimates of the absolute Δ and relative ε errors for various initial distributions and coefficients of variation. The legend of the figure determines the line type for errors with Γ and GW repair time distributions. As $R_{th}(t)$, we again consider the estimate of the reliability function obtained by DEM method.

According to the obtained graphical results, the maximum absolute error among the considered cases is $\max(\Delta) < 0.005$. The maximum value of the relative error varies in the interval is $\max(\varepsilon) \approx [0.003, 0.014]$. Given the given values $N = 10^5$ and $T = 10^5$, these results are objectively satisfactory.

Figure 8 shows the graphs of the reliability function for Γ and GW distributions of elements' lifetime. The experiment demonstrates results similar to those in the previous example. The value of the coefficient of variation of repair time $v_b = 5$ leads to an increase in the values of the reliability function on the considered interval t. In addition, the shape of repair time distribution affects the behavior of this curve as $v_b > 1$. It is obvious that the reliability function curves are quite close to each other for different initial distributions, but fixed values of the parameters μ_a and v_a.

Table 3 presents the mean system lifetime, including estimates obtained by DEM method. Tables 3a and 3b demonstrate close results for different components' lifetime distributions. The mean system lifetime μ_R differs only at the second or third decimal places. Thus, we can conclude that the mean system lifetime is insensitive to the shape of lifetime distribution for fixed mean μ_a and coefficient of variation v_a.

Table 3. Mean lifetime of $< GI_2|GI|1 >$ system

$B(\cdot)$, v_b	DEM	Algorithm
Γ, $v_b = 0.5$	1.4059	1.4003
Γ, $v_b = 5$	4.1305	4.2208
GW, $v_b = 0.5$	1.3900	1.3877
GW, $v_b = 5$	2.6242	2.6349

(a) $A \sim \Gamma,\ v_a = 0.5$

$B(\cdot)$, v_b	DEM	Algorithm
Γ, $v_b = 0.5$	1.4061	1.4044
Γ, $v_b = 5$	4.1637	4.2347
GW, $v_b = 0.5$	1.3983	1.3985
GW, $v_b = 5$	2.6329	2.6433

(b) $A \sim GW,\ v_a = 0.5$

Additionally, comparing the values in Tables 2 and 3, we can conclude that the mean lifetime of the system has low sensitivity to the shape of components' lifetime distribution for fixed mean μ_a and coefficient of variation $v_a \leqslant 1$.

7. CONCLUSION

The article introduces the concept of the Marked Markov process, which is used to study a repairable hot double redundant system with arbitrary initial distributions. A method and algorithm for calculating the reliability function of such a system are proposed. The results obtained using the new approach were validated by comparing them with the results of analytical calculations and discrete-event modeling. The new approach showed high accuracy in comparison with known methods.

The main advantage of the proposed modeling algorithm compared to discrete-event modeling is that the presented procedure is entirely based on a mathematically formulated approach and, in essence, consists of implementing marks' transformation that uniquely determines the system operation. Moreover, this algorithm eliminates the need to write additional code for the discrete-event modeling program, which significantly simplifies the process of studying a repairable hot double redundant system with arbitrary initial distributions.

In the future, it is planned to apply this approach to the study of other complex stochastic systems, including the study of the reliability characteristics of a k-out-of-n system with arbitrary initial distributions and an arbitrary number of repair units.

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